Diffraction by an imperfect half plane in a bianisotropic medium

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[1] A general theory to study the electromagnetic diffraction by imperfect half planes immersed in linear homogeneous bianisotropic media is presented. The problem is formulated in terms of Wiener-Hopf equations by deriving explicit spectral domain expressions for the characteristic impedances of bianisotropic media, which allow one to exploit their analytical properties. In the simpler case of perfect electric conducting and perfect magnetic conducting half planes, the Wiener-Hopf equations involve matrices of order 2, which can be factorized in closed form if the constitutive tensors of the bianisotropic material are of special form. Four of these special cases are discussed in detail. In order to deal with the more general problem, a technique to numerically factorize the Wiener-Hopf matrix kernels is presented. Our numerical approach is discussed on one example, by considering the previously unsolved problem of a perfect electric conducting half plane in a gyrotropic medium. The reported numerical results show that the diffracted field contribution is obtained by use of the saddle point integration method.

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1. Introduction

[2] The first rigorous studies of the diffraction by a perfect electric conducting (PEC) half plane immersed in a homogeneous isotropic medium are by *Poincaré* [1892] and *Sommerfeld* [1896]. Noticeable progress in the solution of these problems is attributed to the introduction and use of the Sommerfeld-Malyuzhinets (SM) technique and the Wiener-Hopf (WH) technique [*Senior*, 1978; *Hurd and Luneburg*, 1985; *Budaev*, 1995; *Senior and Volakis*, 1995; *Lüneburg and Serbest*, 2000; *Daniele*, 2003; *Antipov and Silvestrov*, 2006; *Lyalinov and Zhu*, 2006; *Daniele and Lombardi*, 2006]. In particular, we observe that the WH technique can deal with the most general problem of a half plane immersed in an anisotropic or bianisotropic medium, whereas the SM method cannot deal with it, as yet.

[3] The purpose of this paper is to establish a general WH theory to solve the electromagnetic problem of an imperfect half plane immersed in an arbitrary linear medium. Our theory yields to the factorization of matrices of order 2 and 4 in case of perfect and imperfect half plane, respectively. The solution of this kind of problems is known in closed form only for PEC or perfectly magnetic conducting (PMC) half planes surrounded by rather simple media; the perfectly conducting case is simpler because it requires the factorization of scalar kernels [*Seshadri and Rajagopal*, 1963; *Jull*, 1964; *Przezdziecki*, 2000] or of kernel matrices of order 2 [*Hurd and Przezdziecki*, 1981, 1985].

[4] To deal with the most general case, it is therefore convenient to introduce and use approximate techniques. For example, a general approximate factorization method has been introduced by *Daniele* [2004a, 2004b] and *Daniele and Lombardi* [2007], and the impenetrable half plane and wedge problems have been solved very efficiently by approximate factorization by *Daniele and Lombardi* [2006]. In this paper, this factorization method is applied to effectively solve a previously unsolved problem, thereby showing new results for a PEC half plane in a gyrotropic medium.

[5] In this connection we observe that problems involving very complex algebraic manipulations, such as those required by the problems of this paper as well as by those of *Graglia et al.* [1991], can nowadays be solved because powerful algebraic manipulator codes are readily available. The results of this paper were obtained by intensive use of the computing software Mathematica®.

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In fact, although the matrix formulas of this paper may look rather simple, the general explicit expression of each matrix coefficient in terms of the electromagnetic parameters usually occupies several pages.

[6] To facilitate the reading and the comprehension of the material presented in this paper, section 2 considers in detail the simpler cases of a PEC and of a PMC half plane; the most general case of an imperfect half plane is reported in Appendix A. The characteristic impedances and admittances of the bianisotropic medium that surrounds the half plane are defined in section 3, where we also provide two different methods for their evaluation. Special cases of PEC and PMC half planes amenable to closed-form solution are then discussed in section 4, whereas the general method to numerically factorize the WH kernels is given in section 5. One example of application of our numerical factorization method is also discussed in section 5, thereby showing with numerical results that the diffracted field contribution can be obtained by the saddle point integration method.

2. Wiener-Hopf Formulation of the Half Plane Problem

[7] We consider the frequency domain diffraction problem of a plane wave impinging on an imperfect half plane immersed in a homogeneous bianisotropic medium. With reference to Figure 1, the y axis of the Cartesian reference frame $\{z, x, y\}$ is normal to the half plane surface $\{x < 0, y = 0\}$, whereas the z axis lies along the half plane edge. The space-time dependence factor of the incident electromagnetic wave is

$$\exp\left(j\omega t\right)\ \exp\left[-jk_o(\overline{\alpha}_o z + \overline{\eta}_o x + \overline{\kappa}_o y)\right] \tag{1}$$

where ω is the angular frequency, t the time, and k_o the free-space wave number. Notice that the unnormalized components of the vector wave number of the incident wave are

$$\alpha_o = k_o\overline{\alpha}_o, \quad \eta_o = k_o\overline{\eta}_o, \quad \kappa_o = k_o\overline{\kappa}_o \tag{2}$$

and that the time dependence factor $\exp(j\omega\, t)$ is assumed and suppressed throughout the paper, whereas for phase continuity, the z dependence factor $\exp(-j\alpha_0 z)$ is common to all incident and diffracted field components.

[8] In the frequency domain, the constitutive relations of the linear homogenous medium in which the half plane is immersed are [*Graglia et al.*, 1991]

$$\begin{cases} \mathbf{D} = \varepsilon_o\left(\overline{\boldsymbol{\varepsilon}}\,\mathbf{E} + \overline{\boldsymbol{\xi}}\,Z_o\,\mathbf{H}\right) \\ \mathbf{B} = \mu_o\left(\overline{\boldsymbol{\mu}}\,\mathbf{H} + \overline{\boldsymbol{\zeta}}\,Y_o\,\mathbf{E}\right) \end{cases} \tag{3}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ the magnetic field, $\mathbf{D}$ and $\mathbf{B}$ the electric and magnetic flux densities, respectively; ε_o is the electric permittivity and μ_o the magnetic permeability of free space; Z_o and $Y_o = 1/Z_o$ is the free space impedance and admittance, respectively; $\overline{\boldsymbol{\xi}}$ and $\overline{\boldsymbol{\zeta}}$ are cross tensors that relate the magnetic and electric field to the electric and magnetic flux densities, respectively. The dimensionless overlined tensors appearing in (3) define the dimensioned constitutive tensors

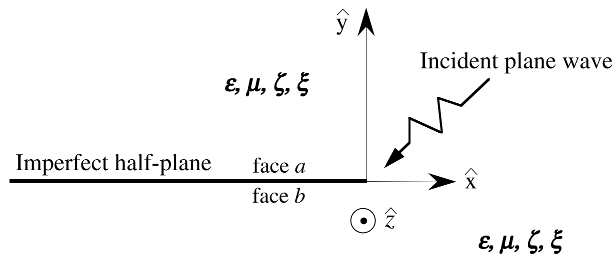

Figure 1. Geometry of the problem.

$$\boldsymbol{\varepsilon} = \varepsilon_o\,\overline{\boldsymbol{\varepsilon}}, \quad \boldsymbol{\mu} = \mu_o\,\overline{\boldsymbol{\mu}}, \quad \boldsymbol{\xi} = \frac{k_o}{\omega}\,\overline{\boldsymbol{\xi}}, \quad \boldsymbol{\zeta} = \frac{k_o}{\omega}\,\overline{\boldsymbol{\zeta}} \tag{4}$$

[9] In the rectangular Cartesian coordinates $\{z, x, y\}$ all vector quantities may be written as three-element column vectors; then, the constitutive tensors are written as (3×3) matrices. To compact the notation, in the following, the $(n \times n)$ identity matrix is indicated by $\overline{\mathbf{1}}_n$.

[10] For lossless media, the following conditions hold [*Kong*, 1975]:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^+, \quad \boldsymbol{\mu} = \boldsymbol{\mu}^+, \quad \boldsymbol{\zeta} = \boldsymbol{\xi}^+ \tag{5}$$

where the superscript plus is used to denote the transpose and complex conjugate operation.

[11] The boundary conditions for the imperfect half plane ($\forall z,\ x < 0,\ y = 0$) in terms of the tangential components $\mathbf{E}_t = \hat{\mathbf{z}}E_z + \hat{\mathbf{x}}E_x$ and $\mathbf{H}_t = \hat{\mathbf{z}}H_z + \hat{\mathbf{x}}H_x$ of the electric and magnetic field on the upper ($y = 0_+$) and lower ($y = 0_-$) half plane face are

$$\begin{aligned} \hat{\mathbf{y}} \times \mathbf{E}_t(x, 0_+, z) &= -\mathbf{Z}_a\mathbf{H}_t(x, 0_+, z) - \mathbf{Z}_{ab}\mathbf{H}_t(x, 0_-, z) \\ \hat{\mathbf{y}} \times \mathbf{E}_t(x, 0_-, z) &= \mathbf{Z}_b\mathbf{H}_t(x, 0_-, z) + \mathbf{Z}_{ba}\mathbf{H}_t(x, 0_+, z) \end{aligned} \tag{6}$$

where the (2×2) impedance matrices $\mathbf{Z}_a$, $\mathbf{Z}_b$, $\mathbf{Z}_{ab}$, and $\mathbf{Z}_{ba}$ depend on the material of the half plane; this is impenetrable if its faces are decoupled, that is for $\mathbf{Z}_{ab} = \mathbf{Z}_{ba} = \mathbf{0}$. Notice that the boundary condition valid for a PMC half plane ($\forall z,\ x < 0,\ y = 0$)

$$\mathbf{H}_t(x, 0, z) = \mathbf{0} \tag{7}$$

usually poses simpler problems; the solutions of PMC problems are normally obtained by duality from those of the PEC problems. The boundary condition valid for a PEC half plane ($\forall z$, $x < 0$, $y = 0$) is

$$\mathbf{E}_t(x,0,z) = \mathbf{0} \tag{8}$$

[12] To formulate the WH problem, we introduce the one-dimensional Fourier transforms

$$\begin{cases} \mathbf{V}(\eta,y) = \exp(j\alpha_0 z)\int_{-\infty}^{+\infty} \hat{\mathbf{y}} \times \mathbf{E}_t(x,y,z)\exp(j\eta x)\mathrm{d}x \\ \mathbf{I}(\eta,y) = \exp(j\alpha_0 z)\int_{-\infty}^{+\infty} \mathbf{H}_t(x,y,z)\exp(j\eta x)\mathrm{d}x \end{cases} \tag{9}$$

and define

$$\begin{cases} \mathbf{V}_{ab} = \mathbf{V}(\eta,0_\pm) \\ \mathbf{I}_{ab} = \pm\mathbf{I}(\eta,0_\pm) \end{cases} \tag{10}$$

According to the uniqueness theorem, the knowledge of the tangential field $\mathbf{H}_t$ on a closed surface permits one to obtain the tangential electric field $\mathbf{E}_t$, and viceversa. By considering the whole $y = 0$ plane as a closed surface that bounds the homogeneous upper half-space, and the lower half-space, one can prove that the Fourier transforms are related by the following algebraic equations:

$$\mathbf{V}_a = \overrightarrow{\mathbf{Z}}\,\mathbf{I}_a, \quad \mathbf{I}_a = \overrightarrow{\mathbf{Y}}\,\mathbf{V}_a, \tag{11}$$

$$\mathbf{V}_b = \overleftarrow{\mathbf{Z}}\,\mathbf{I}_b, \quad \mathbf{I}_b = \overleftarrow{\mathbf{Y}}\,\mathbf{V}_b \tag{12}$$

where $\overrightarrow{\mathbf{Z}}$, $\overleftarrow{\mathbf{Z}}$ (and $\overrightarrow{\mathbf{Y}}$, $\overleftarrow{\mathbf{Y}}$) are (2×2) matrices that represent the impedance (admittance) of the upper ($y > 0$) and of the lower ($y < 0$) half-space, respectively. The coefficients of these matrices are functions of the spectral variable η, and depend on the electromagnetic properties of the linear medium that surrounds the imperfect half plane. Equations (11) and (12) extend the characteristic impedance concept to an arbitrary indefinite linear medium. The procedure to evaluate the characteristic impedance matrices is explained in section 3. To facilitate the reader, in sections 2.1 and 2.2 we consider in detail only the simpler case of a perfectly conducting half plane; the WH equations for the imperfect half plane are reported in Appendix A.

2.1. WH Equations for Perfectly Conducting Half Planes

[13] In the case of a PEC half plane, the boundary condition (8) yields $\mathbf{V}_a = \mathbf{V}_b = \mathbf{V}_+(\eta)$. By summing $\mathbf{I}_a$ and $\mathbf{I}_b$ (i.e., equations (11) and (12)) one gets

$$\left(\overleftarrow{\mathbf{Y}} + \overrightarrow{\mathbf{Y}}\right)\mathbf{V}_+(\eta) = \mathbf{A}_-(\eta) \tag{13}$$

$$\mathbf{A}_-(\eta) = \mathbf{I}_a + \mathbf{I}_b \tag{14}$$

where $\mathbf{A}_-(\eta)$ is the Fourier transform of the total electric current induced on the half plane.

[14] In the case of a PMC half plane, the boundary condition (7) yields $\mathbf{I}_a = -\mathbf{I}_b = \mathbf{I}_+(\eta)$. By subtracting $\mathbf{V}_a$ and $\mathbf{V}_b$ (i.e., equations (11) and (12)) one gets

$$\left(\overleftarrow{\mathbf{Z}} + \overrightarrow{\mathbf{Z}}\right)\mathbf{I}_+(\eta) = \mathbf{M}_-(\eta) \tag{15}$$

$$\mathbf{M}_-(\eta) = \mathbf{V}_a - \mathbf{V}_b \tag{16}$$

where $\mathbf{M}_-(\eta)$ is the Fourier transform of the total magnetic current induced on the half plane.

[15] For sake of brevity we omit, for a moment, a detailed discussion of the incident wave and simply assume an incident plane wave that, at $y = 0$, yields (see (1))

$$\begin{bmatrix} \hat{\mathbf{y}} \times \mathbf{E}_t^i(x,0,z) \\ \mathbf{H}_t^i(x,0,z) \end{bmatrix} = E_0 \exp(-j\eta_0 x)\exp(-j\alpha_0 z)\begin{bmatrix} \mathbf{e}_{0t} \\ \mathbf{h}_{0t} \end{bmatrix} \tag{17}$$

where $\mathbf{e}_0$ and $\mathbf{h}_0$ are incident polarization vectors, and where the factor $\exp(-j\eta_o x)$ introduces, in the Fourier domain, a pole at $\eta = \eta_o$. The residues of this pole for $\mathbf{V}_+(\eta)$ and $\mathbf{I}_+(\eta)$ are known since they represent, in the Fourier domain, the contribution at $y = 0$ of the incident plane wave. From equations (9) and (17) one gets

$$\mathrm{Res}[\mathbf{V}_+(\eta)]|_{\eta=\eta_o} = jE_o\mathbf{e}_{0t} \tag{18}$$

$$\mathrm{Res}[\mathbf{I}_+(\eta)]|_{\eta=\eta_o} = jE_o\mathbf{h}_{0t} \tag{19}$$

which, together with equations (13) and (15), yield

$$\mathbf{A}_-(\eta) = \mathbf{A}^s_-(\eta) + \left(\overleftarrow{\mathbf{Y}}(\eta_o) + \overrightarrow{\mathbf{Y}}(\eta_o)\right)\frac{jE_o\mathbf{e}_{0t}}{\eta - \eta_o} \tag{20}$$

$$\mathbf{M}_-(\eta) = \mathbf{M}^s_-(\eta) + \left(\overleftarrow{\mathbf{Z}}(\eta_o) + \overrightarrow{\mathbf{Z}}(\eta_o)\right)\frac{jE_o\mathbf{h}_{0t}}{\eta - \eta_o} \tag{21}$$

with $\mathbf{A}^s_-(\eta)$ and $\mathbf{M}^s_-(\eta)$ regular at $\eta = \eta_o$. By substituting these latter equations into (13) and (15) one obtains the nonhomogeneous WH equations for the PEC and the PMC problem. These equations, as well as those reported in Appendix A, always take the following form:

$$\mathbf{G}(\eta)\mathbf{F}_+(\eta) = \mathbf{F}^s_-(\eta) + \frac{\mathbf{R}}{\eta - \eta_0} \tag{22}$$

2.2. Solution of the Nonhomogeneous Equations

[16] The factorization of the matrix kernel $\mathbf{G}(\eta) = \mathbf{G}_-(\eta)\mathbf{G}_+(\eta)$ leads to [see *Daniele*, 2004a]

$$\mathbf{F}_+(\eta) = \mathbf{G}_+^{-1}(\eta)\,\mathbf{G}_-^{-1}(\eta_0)\,\frac{\mathbf{R}_o}{\eta - \eta_0} \tag{23}$$

$$\mathbf{F}_-(\eta) = \mathbf{F}_-^s(\eta) + \frac{\mathbf{R}_o}{\eta - \eta_0} = \mathbf{G}_-(\eta)\,\mathbf{G}_-^{-1}(\eta_0)\,\frac{\mathbf{R}_o}{\eta - \eta_0} \tag{24}$$

[17] In section 4 we illustrate four cases where closed-form factorization is possible. The technique to obtain the factorization in the general case is given in section 5, where we also apply this technique to obtain the diffraction coefficients for a PEC half plane surrounded by a gyrotropic medium, that is, an important, previously unsolved problem.

3. Half-Space Characteristic Impedances and Admittances

[18] The transverse field equations obtained by using the Bresler-Marcuvitz formalism [*Bresler and Marcuvitz*, 1956; *Daniele*, 1971, 2006] yield

$$-\frac{\mathrm{d}}{\mathrm{d}y}\begin{bmatrix}\mathbf{V}\\ \mathbf{I}\end{bmatrix} = \mathbf{P}\begin{bmatrix}\mathbf{V}\\ \mathbf{I}\end{bmatrix} \tag{25}$$

with $\mathbf{V}$ and $\mathbf{I}$ defined in (9) and where $\mathbf{P}$ is a (4 × 4) matrix partitioned into four (2 × 2) submatrices

$$\mathbf{P} = \begin{bmatrix}\mathbf{T}_e(\eta) & \mathbf{Z}(\eta)\\ \mathbf{Y}(\eta) & \mathbf{T}_h(\eta)\end{bmatrix} \tag{26}$$

$\mathbf{P}(\eta)$, $\mathbf{T}_e(\eta)$, $\mathbf{Z}(\eta)$, $\mathbf{Y}(\eta)$, and $\mathbf{T}_h(\eta)$ are second-degree polynomial matrices, since all their coefficients are second degree polynomials of the variable η. The general expression of $\mathbf{P}(\eta)$ can be obtained as reported in Appendix B.

[19] We now discuss two methods to evaluate the characteristic impedances $\overrightarrow{\mathbf{Z}}$ and $\overleftarrow{\mathbf{Z}}$ introduced in (11) and (12) by first observing that the coefficients of the matrix

$$\overline{\mathbf{P}} = \begin{bmatrix}\dfrac{\mathbf{T}_e(\eta)}{k_o} & \dfrac{\mathbf{Z}(\eta)}{k_o Z_o}\\ \dfrac{\mathbf{Y}(\eta)}{k_o Y_o} & \dfrac{\mathbf{T}_h(\eta)}{k_o}\end{bmatrix} \tag{27}$$

$$-\frac{\mathrm{d}}{\mathrm{d}(k_o y)}\begin{bmatrix}\mathbf{V}\\ Z_o\mathbf{I}\end{bmatrix} = \overline{\mathbf{P}}\begin{bmatrix}\mathbf{V}\\ Z_o\mathbf{I}\end{bmatrix} \tag{28}$$

are dimensionless. Notice that $\mathbf{P}$ and $\overline{\mathbf{P}}$ have the same eigenvectors, and that the eigenvalues of $\mathbf{P}$ are k_o times those of $\overline{\mathbf{P}}$.

3.1. First Evaluation Method

[20] The general solution of (25) in terms of the four eigenvectors γ_i and eigenvalues $[V_z^{(i)}, V_x^{(i)}, I_z^{(i)}, I_x^{(i)}]^t$ of the matrix $\mathbf{P}$ reads

$$\begin{bmatrix}\mathbf{V}\\ \mathbf{I}\end{bmatrix} = \begin{bmatrix}V_z\\ V_x\\ I_z\\ I_x\end{bmatrix} = \sum_{i=1}^{4} C_i \exp(-\gamma_i y)\begin{bmatrix}V_z^{(i)}\\ V_x^{(i)}\\ I_z^{(i)}\\ I_x^{(i)}\end{bmatrix} \tag{29}$$

where the scalar coefficients C_i, for i = 1, 4, are y-independent. If the medium surrounding the half plane is passive, one can suppose that two eigenvalues (γ_1 and γ_2) have a nonnegative real part, whereas γ_3 and γ_4 have a nonpositive real part. A proof of this conjecture, that is verified by all the passive media we have considered in our studies, is discussed by *Daniele* [2006]. In the lower half-space, the solution is obtained by taking $C_1 = C_2 = 0$, whereas one sets $C_3 = C_4 = 0$ in the upper $y > 0$ half-space. Thus, for $y > 0$, one has

$$\mathbf{V} = \begin{bmatrix}V_z^{(1)} & V_z^{(2)}\\ V_x^{(1)} & V_x^{(2)}\end{bmatrix}\begin{bmatrix}C_1\exp(-\gamma_1 y)\\ C_2\exp(-\gamma_2 y)\end{bmatrix} \tag{30}$$

$$\mathbf{I} = \begin{bmatrix}I_z^{(1)} & I_z^{(2)}\\ I_x^{(1)} & I_x^{(2)}\end{bmatrix}\begin{bmatrix}C_1\exp(-\gamma_1 y)\\ C_2\exp(-\gamma_2 y)\end{bmatrix} \tag{31}$$

which, together with (10) and (11), yield

$$\overrightarrow{\mathbf{Z}} = \begin{bmatrix}V_z^{(1)} & V_z^{(2)}\\ V_x^{(1)} & V_x^{(2)}\end{bmatrix}\begin{bmatrix}I_z^{(1)} & I_z^{(2)}\\ I_x^{(1)} & I_x^{(2)}\end{bmatrix}^{-1} \tag{32}$$

Similarly, in the indefinite lower medium ($y < 0$) one gets (see (10) and (12))

$$\overleftarrow{\mathbf{Z}} = \begin{bmatrix}V_z^{(3)} & V_z^{(4)}\\ V_x^{(3)} & V_x^{(4)}\end{bmatrix}\begin{bmatrix}I_z^{(3)} & I_z^{(4)}\\ I_x^{(3)} & I_x^{(4)}\end{bmatrix}^{-1} \tag{33}$$

3.2. Second Evaluation Method

[21] The expressions of the characteristic impedances (32) and (33) are in general, too complex to permit one to study their properties, in the complex η plane. More convenient expressions are obtained by eliminating $\mathbf{I}$ or $\mathbf{V}$ from (25) by use of (26):

$$\begin{aligned}\frac{\mathrm{d}^2\mathbf{V}}{\mathrm{d}y^2} - \mathbf{A}_V\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}y} + \mathbf{B}_V\,\mathbf{V} = \mathbf{0}\\ \frac{\mathrm{d}^2\mathbf{I}}{\mathrm{d}y^2} - \mathbf{A}_I\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}y} + \mathbf{B}_I\,\mathbf{I} = \mathbf{0}\end{aligned} \tag{34}$$

and by assuming solutions of the form

$$\mathbf{V} = \exp(-\boldsymbol{\gamma}_V y)\mathbf{V}_o, \quad \mathbf{I} = \exp(-\boldsymbol{\gamma}_I y)\mathbf{I}_o \tag{35}$$

to obtain

$$\begin{aligned} \boldsymbol{\gamma}_V^2 + \mathbf{A}_V \boldsymbol{\gamma}_V + \mathbf{B}_V &= \mathbf{0} \\ \boldsymbol{\gamma}_I^2 + \mathbf{A}_I \boldsymbol{\gamma}_I + \mathbf{B}_I &= \mathbf{0} \end{aligned} \tag{36}$$

with

$$\begin{bmatrix} \mathbf{A}_V \\ \mathbf{B}_V \end{bmatrix} = -\begin{bmatrix} \mathbf{T}_e + \mathbf{Z}\mathbf{T}_h\mathbf{Z}^{-1} \\ \mathbf{Z}\left(\mathbf{Y} - \mathbf{T}_h\mathbf{Z}^{-1}\mathbf{T}_e\right) \end{bmatrix} \tag{37}$$

$$\begin{bmatrix} \mathbf{A}_I \\ \mathbf{B}_I \end{bmatrix} = -\begin{bmatrix} \mathbf{T}_h + \mathbf{Y}\mathbf{T}_e\mathbf{Y}^{-1} \\ \mathbf{Y}\left(\mathbf{Z} - \mathbf{T}_e\mathbf{Y}^{-1}\mathbf{T}_h\right) \end{bmatrix} \tag{38}$$

In general, the matrix equations (36) admit several solutions; for a passive medium we choose those with real positive ($\overrightarrow{\boldsymbol{\gamma}}_V$, $\overrightarrow{\boldsymbol{\gamma}}_I$) and real negative ($\overleftarrow{\boldsymbol{\gamma}}_V$, $\overleftarrow{\boldsymbol{\gamma}}_I$) eigenvalues for $y > 0$ and $y < 0$, respectively. From equations (25) and (26), one gets

$$\begin{aligned} \boldsymbol{\gamma}_V \mathbf{V} &= \mathbf{T}_e \mathbf{V} + \mathbf{Z}\mathbf{I} \\ \boldsymbol{\gamma}_I \mathbf{I} &= \mathbf{Y}\mathbf{V} + \mathbf{T}_h \mathbf{I} \end{aligned} \tag{39}$$

which immediately yields

$$\begin{aligned} \overrightarrow{\mathbf{Z}} &= \mathbf{Y}^{-1}\left(\overleftarrow{\boldsymbol{\gamma}}_I - \mathbf{T}_h\right), \quad \overleftarrow{\mathbf{Z}} = \mathbf{Y}^{-1}\left(\overleftarrow{\boldsymbol{\gamma}}_I - \mathbf{T}_h\right) \\ \overrightarrow{\mathbf{Y}} &= \mathbf{Z}^{-1}\left(\overleftarrow{\boldsymbol{\gamma}}_V - \mathbf{T}_e\right), \quad \overleftarrow{\mathbf{Y}} = \mathbf{Z}^{-1}\left(\overleftarrow{\boldsymbol{\gamma}}_V - \mathbf{T}_e\right) \end{aligned} \tag{40}$$

Although (32) and (33) differ from (40), these expressions are equivalent and yield the same numerical results in all cases we have considered.

3.3. Evaluation of the Eigenvalues

[22] The eigenvalues of $\boldsymbol{\gamma}_V$ and $\boldsymbol{\gamma}_I$ are the same; these eigenvalues are γ_1 and γ_2 (for $\overrightarrow{\boldsymbol{\gamma}}_V$ and $\overrightarrow{\boldsymbol{\gamma}}_I$), and γ_3 and γ_4 (for $\overleftarrow{\boldsymbol{\gamma}}_V$ and $\overleftarrow{\boldsymbol{\gamma}}_I$). In the most general case of a linear bianisotropic medium, one has to solve the matrix equation (36) which, by omitting the V and I subscripts, reads

$$\boldsymbol{\gamma}^2 + \mathbf{A}\boldsymbol{\gamma} + \mathbf{B} = \mathbf{0} \tag{41}$$

Since this equation involves only (2×2) matrices, it is associated with the characteristic equation

$$\boldsymbol{\gamma}^2 - t_\gamma \boldsymbol{\gamma} + \Delta_\gamma \mathbf{1} = \mathbf{0} \tag{42}$$

where the scalar coefficients t_γ and Δ_γ are the trace and the determinant of $\boldsymbol{\gamma}$, respectively; these coefficients are

$$t_\gamma = \gamma_1 + \gamma_2, \quad \Delta_\gamma = \gamma_1 \gamma_2 \tag{43}$$

for the upper half-space $y > 0$, and

$$t_\gamma = \gamma_3 + \gamma_4, \quad \Delta_\gamma = \gamma_3 \gamma_4 \tag{44}$$

for the lower half-space $y < 0$. The problem posed by (41) is difficult whenever $\mathbf{A}$ and $\mathbf{B}$ are of arbitrary order and not commutative. Fortunately, in our case, equations (41) and (42) allows one to eliminate $\boldsymbol{\gamma}^2$ and yield

$$\boldsymbol{\gamma} = \left(\mathbf{A} + t_\gamma \mathbf{1}\right)^{-1}\left(\Delta_\gamma \mathbf{1} - \mathbf{B}\right) \tag{45}$$

Now, according to Cayley's theorem, one has $(\mathbf{A} + t_\gamma \mathbf{1})^{-1} = x\mathbf{1} + y\mathbf{A}$ and from (45),

$$\boldsymbol{\gamma} = x\Delta_\gamma \mathbf{1} - x\mathbf{B} + y\Delta_\gamma \mathbf{A} - y\mathbf{A}\mathbf{B} \tag{46}$$

with

$$\begin{aligned} x &= \frac{t_\gamma + t_A}{t_\gamma\left(t_\gamma + t_A\right) + \Delta_A} \\ y &= \frac{-1}{t_\gamma\left(t_\gamma + t_A\right) + \Delta_A} \end{aligned} \tag{47}$$

where t_A and Δ_A are the trace and the determinant of $\mathbf{A}$, respectively.

[23] The characteristic impedances and admittances are obtained directly from (40) after evaluating and ordering the four eigenvalues of the matrix $\overline{\mathbf{P}}$ given in (27)

$$\det\left[\overline{\gamma}\,\mathbf{1}_4 - \overline{\mathbf{P}}\right] = \overline{\gamma}^4 + a\overline{\gamma}^3 + b\overline{\gamma}^2 + c\overline{\gamma} + d = 0 \tag{48}$$

The results provided by (48) always coincide with those of *Graglia et al.* [1991]; the compact expression of the four solutions of (48), for $i = 1, 4$, is

$$\overline{\gamma}_i = -\frac{a}{4} \pm \frac{1}{2}\left(\sqrt{T} \pm \sqrt{Q}\right) \tag{49}$$

with

$$T = \frac{a^2}{4} + \frac{\left(b^2 - 3ac + 12d\right)}{3u^{1/3}} + \frac{u^{1/3} - 2b}{3} \tag{50}$$

$$Q = \frac{3a^2}{4} - 2b - T + \frac{a^3 - 4ab + 8c}{4\sqrt{T}} \tag{51}$$

$$u = \frac{v + \sqrt{s}}{2} \tag{52}$$

$$v = 2b^3 + 9\left(3a^2 d - abc + 3c^2 - 8bd\right) \tag{53}$$

$$s = v^2 - 4\left(b^2 - 3ac + 12d\right)^3 \tag{54}$$

In this connection, it is of importance to point out that the quantities $\mathbf{T}_e(\eta)/k_o$, $\mathbf{T}_h(\eta)/k_o$, $\mathbf{Z}(\eta)/(k_oZ_o)$, and $\mathbf{Y}(\eta)/(k_oY_o)$ appearing into (27) are independent from k_o, Z_o and Y_o. Furthermore, these matrices result to be independent from the angular frequency ω if the dimensionless tensors $\overline{\varepsilon}$, $\overline{\boldsymbol{\mu}}$, $\overline{\boldsymbol{\xi}}$, and $\overline{\boldsymbol{\zeta}}$ do not depend on ω. Thus, by considering equations (40) and the normalized version of (46), one immediately recognizes that the normalized propagation matrices $\boldsymbol{\gamma}_V/k_o$ and $\boldsymbol{\gamma}_I/k_o$, and the normalized characteristic impedance and admittance matrices $\overrightarrow{\mathbf{Z}}/Z_o$, $\overleftarrow{\mathbf{Z}}/Z_o$, $\overrightarrow{\mathbf{Y}}/Y_o$, and $\overleftarrow{\mathbf{Y}}/Y_o$ are independent from k_o, Z_o and Y_o. The main consequence of this behavior is that the eigenvectors of the normalized propagation matrices are equal to those of the unnormalized ones ($\boldsymbol{\gamma}_V$ and $\boldsymbol{\gamma}_I$), and that the eigenvalues of $\overline{\boldsymbol{\gamma}}_V = \boldsymbol{\gamma}_V/k_o$ and $\overline{\boldsymbol{\gamma}}_I = \boldsymbol{\gamma}_I/k_o$ are independent from k_o, Z_o and Y_o.

[24] By setting, as suggested by equation (2),

$$\eta = k_o\,\overline{\eta}, \quad \alpha_o = k_o\,\overline{\alpha}_o, \quad \overline{\gamma} = j\overline{\kappa} \tag{55}$$

into equation (48), one obtains an algebraic equation of the form $f(\overline{\eta}, \overline{\kappa}) = 0$ whose coefficient do not depend on k_o, Z_o, and Y_o. The true physical eigenvalues γ_i, for i = 1, 4, of the $\mathbf{P}$ matrix are

$$\gamma_i = jk_o\,\overline{\kappa}_i \tag{56}$$

where $\overline{\kappa}_i$ are suitable solutions of the equation $f(\overline{\eta}, \overline{\kappa}) = 0$. It can be proven that $f(\overline{\eta}, \overline{\kappa})$ is an algebraic function of order 4 both in $\overline{\eta}$ and $\overline{\kappa}$.

[25] In order to appreciate the physical meaning of the function $f(\overline{\eta}, \overline{\kappa})$, let us consider a linear lossless material and its two dispersion surfaces in the $\{\alpha, \eta, \kappa\}$ space. In the present paper, α is constant, since we consider a problem invariant in the z direction. In fact, the value $\alpha = \alpha_o$ appears in the exponential factor $\exp(-j\alpha_o z)$ that expresses the z dependence of the electromagnetic fields. In the $(\overline{\eta}, \overline{\kappa})$ plane, the solutions of the equation $f(\overline{\eta}, \overline{\kappa}) = 0$ lie on two curves obtained by cutting the dispersion surfaces with the $\alpha = \alpha_o$ plane. Typical dispersion curves for a lossless medium are shown in Figure 2, where four real values of $\overline{\kappa}$ (and consequently of $\gamma = jk_o\overline{\kappa}$) are associated with a given value of $\overline{\eta}$. The two solutions that hold in the $y < 0$ half-space are those with $\overline{\kappa} < 0$, the remaining two with $\overline{\kappa} > 0$ are valid in the $y > 0$ half-space.

[26] To apply the WH technique, one has to know the properties of the multivalued functions $\gamma_i(\overline{\eta}) = jk_o\overline{\kappa}_i(\overline{\eta})$. By noticing that the coefficient of $\overline{\gamma}^4$ in (48) is equal to one and by considering $f(\overline{\eta}, \overline{\kappa})$ a function of the variable $\overline{\kappa}$, the branch points of the functions γ_i in the $\overline{\eta}$ plane belong to the set of points defined by all the zeroes of the discriminant $D(\overline{\eta})$ of the polynomial function $f(\overline{\eta}, \overline{\kappa})$

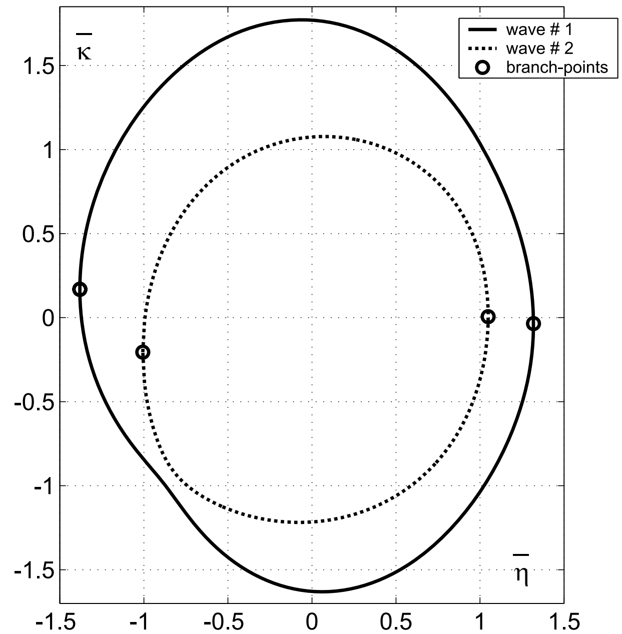

Figure 2. Dispersion curves for a lossless gyrotropic medium with $\overline{\boldsymbol{\mu}} = \mathbf{1}$, and $\overline{\varepsilon}$ as in (117). The curves show the $(\overline{\eta}, \overline{\kappa})$ values for $\overline{\alpha} = \overline{\alpha}_o = 1/2$; the corresponding vector wave numbers are $\mathbf{k} = k_o\,(\overline{\alpha}_o\hat{\mathbf{z}} + \overline{\eta}\,\hat{\mathbf{x}} + \overline{\kappa}\,\hat{\mathbf{y}})$. The four branch points given in (120) and (121) are circled.

[*Bliss*, 2004]. In all the cases we have studied, the zeroes of the discriminant $D(\overline{\eta})$ coincide with the zeroes of the function $s(\overline{\eta})$ given in (54). In general, one has 12 branch points. For lossless media, four of these are real; the $\overline{\eta}$ values of the real branch points are the abscissas of the four vertical lines which are tangent to the two dispersion curves. In Figure 2, the branch points are marked by circles.

[27] As previously noticed, the propagation matrices $\overrightarrow{\gamma}$ (for $y > 0$) and $\overleftarrow{\gamma}$ (for $y < 0$) depend on $(\gamma_1 + \gamma_2)$, $\gamma_1\gamma_2$ and on $(\gamma_3 + \gamma_4)$, $\gamma_3\gamma_4$, respectively. It is remarkable that with the exception of the four real branch points, all the other branch points of γ_1, γ_2, γ_3, and γ_4 do not appear in the quantities $(\gamma_1 + \gamma_2)$, $(\gamma_3 + \gamma_4)$, $\gamma_1\gamma_2$, and $\gamma_3\gamma_4$. This means that the characteristic impedance (and admittance) has only four branch points, that are real for lossless media. Although it is rather simple to prove this property for an $f(\overline{\eta}, \overline{\kappa})$ biquadratic in $\overline{\kappa}$, as it happens for media with vanishing $\mathbf{T}_e(\eta)$ and $\mathbf{T}_h(\eta)$ matrices [*Hurd and Przezdziecki*, 1981], in the most general case this property can only be proved after algebraic computer manipulations, for example by expanding in the neighborhood of the complex branch point $\overline{\eta}_{\mathrm{br}}$ the quantities $(\gamma_1 + \gamma_2)$, $\gamma_1\gamma_2$, $(\gamma_3 + \gamma_4)$, and $\gamma_3\gamma_4$ in terms of Puiseux series. The point $\overline{\eta}_{\mathrm{br}}$ is an ordinary point if, in the Puiseux series, the term containing the factor

$(\overline{\eta} - \overline{\eta}_{\rm br})^{1/r}$, with r integer, is associated to a coefficient numerically equal to zero.

4. Special Cases With Closed-Form Solutions

[28] The general expressions of the characteristic impedances and admittances (32), (33), or (40) simplify in special cases. This section reports four important special cases for which the half plane diffraction problem is solvable in closed form by using the WH approach.

4.1. Isotropic and Bi-isotropic Media

[29] In case of a bi-isotropic (chiral) material defined by the parameters $\boldsymbol{\varepsilon} = \varepsilon\overline{\mathbf{1}}_3$, $\boldsymbol{\mu} = \mu\overline{\mathbf{1}}_3$, $\boldsymbol{\xi} = -j\vartheta\overline{\mathbf{1}}_3$, $\boldsymbol{\zeta} = +j\vartheta\overline{\mathbf{1}}_3$ one gets

$$\overrightarrow{\mathbf{Z}} = \begin{bmatrix} Z_{11}(\alpha_0,\eta) & Z_m(\alpha_0,\eta) \\ Z_m(\alpha_0,\eta) & Z_{22}(\alpha_0,\eta) \end{bmatrix} \tag{57}$$

$$\overleftarrow{\mathbf{Z}} = \begin{bmatrix} Z_{11}(-\alpha_0,\eta) & -Z_m(-\alpha_0,\eta) \\ -Z_m(-\alpha_0,\eta) & Z_{22}(-\alpha_0,\eta) \end{bmatrix} \tag{58}$$

with

$$Z_{11}/p_c = (k_1\,\chi_1 - j\alpha_0\,\eta)(k_2\,\chi_2 + j\alpha_0\,\eta) \tag{59}$$

$$Z_{22}/p_c = \tau_1^2\,\tau_2^2 \tag{60}$$

$$Z_m/p_c = \frac{\alpha_0\,\eta\left(\tau_1^2+\tau_2^2\right) + jk_1\,\chi_1\,\tau_2^2 - jk_2\,\chi_2\,\tau_1^2}{2} \tag{61}$$

$$p_c/Z_o = \frac{k_1+k_2}{k\left[j\alpha_0\,\eta\left(\tau_1^2-\tau_2^2\right) + k_1\,\chi_1\,\tau_2^2 + k_2\,\chi_2\,\tau_1^2\right]} \tag{62}$$

and where

$$Z_o = \sqrt{\frac{\mu}{\varepsilon}} \tag{63}$$

$$k = \omega\sqrt{\varepsilon\mu} \tag{64}$$

$$k_{\substack{1\\2}} = k \pm \omega\vartheta \tag{65}$$

$$\tau_{1,2} = \sqrt{k_{1,2}^2 - \alpha_0^2} \tag{66}$$

$$\chi_{1,2} = \sqrt{\tau_{1,2}^2 - \eta^2} \tag{67}$$

[30] When the chirality factor ϑ vanishes, the characteristic impedances simplify into the expected free-space result:

$$\overrightarrow{\mathbf{Z}} = \overleftarrow{\mathbf{Z}} = \frac{Z_o}{k\sqrt{k^2-\alpha_0^2-\eta^2}}\begin{bmatrix} k^2-\eta^2 & \alpha_0\,\eta \\ \alpha_0\,\eta & k^2-\alpha_0^2 \end{bmatrix} \tag{68}$$

In order to ascertain the possibility to obtain closed form factorizations of the matrix kernels appearing into (13) and (15), or into (A4), it is of importance to know the expression of the characteristic impedance matrices. Notice that the known closed form solutions available in the literature are relative only to the case of a PEC or a PMC half plane; these solutions involve the factorization of (2 × 2) matrices. At skew incidence, that is for $\alpha_0 \neq 0$, the factorization of these matrices is often simplified by the transformation [*Senior*, 1978; *Lüneburg and Serbest*, 2000]

$$\mathbf{t} = \begin{bmatrix} \eta & -\alpha_0 \\ \alpha_0 & \eta \end{bmatrix}, \quad \mathbf{t}_a = \begin{bmatrix} \eta & \alpha_0 \\ -\alpha_0 & \eta \end{bmatrix} \tag{69}$$

For example, in case of a PMC half plane immersed in a chiral medium, one has to factorize the matrix $(\overrightarrow{\mathbf{Z}} + \overleftarrow{\mathbf{Z}})$ (see (15)), with

$$\mathbf{t}\left(\overrightarrow{\mathbf{Z}} + \overleftarrow{\mathbf{Z}}\right)\mathbf{t}_a = \frac{2Z_o(k_1+k_2)\left(\alpha_0^2+\eta^2\right)}{k(k_1\,\chi_2+k_2\,\chi_1)}\begin{bmatrix} \chi_1\,\chi_2 & 0 \\ 0 & k_1k_2 \end{bmatrix} \tag{70}$$

Thus factorization is accomplished by factorizing the scalars $\chi_1\,\chi_2$, and $(k_1\chi_2 + k_2\chi_1)$. Similar considerations apply to the PEC case, exhaustively studied by *Przezdziecki* [2000].

4.2. Bianisotropic Media With Diagonal $\mathbf{T}_e$ and $\mathbf{T}_h$ Matrices

[31] Explicit solutions can also be obtained when the matrix coefficients of (36) commute, as it happens, for example, in case of diagonal $\mathbf{T}_e(\eta) = T_e\,\overline{\mathbf{1}}_2$ and $\mathbf{T}_h(\eta) = T_h\,\overline{\mathbf{1}}_2$. In this case one gets

$$\mathbf{V} = \mathbf{V}_1\,\exp\left(-\overrightarrow{\mathbf{A}}\,y\right) + \mathbf{V}_2\,\exp\left(-\overleftarrow{\mathbf{A}}\,y\right) \tag{71}$$

$$\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{A}}\,\mathbf{Z} \tag{72}$$

$$\overleftarrow{\mathbf{Z}} = \overleftarrow{\mathbf{A}}\,\mathbf{Z} \tag{73}$$

with

$$\overrightarrow{\mathbf{A}} = \left[\frac{(T_e + T_h)\bar{\mathbf{I}}_2 + \sqrt{(T_e - T_h)^2\bar{\mathbf{I}}_2 + 4(\mathbf{ZY} - T_e T_h \bar{\mathbf{I}}_2)}}{2}\right]^{-1} \tag{74}$$

$$\overleftarrow{\mathbf{A}} = \left[\frac{(T_e + T_h)\bar{\mathbf{I}}_2 - \sqrt{(T_e - T_h)^2\bar{\mathbf{I}}_2 + 4(\mathbf{ZY} - T_e T_h \bar{\mathbf{I}}_2)}}{2}\right]^{-1} \tag{75}$$

In spite of the fact that the cases of diagonal $\mathbf{T}_e(\eta)$ and $\mathbf{T}_h(\eta)$ can hardly be classified, we notice that this certainly happens in the special case

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{zz} & \varepsilon_{zx} & 0 \\ \varepsilon_{xz} & \varepsilon_{xx} & 0 \\ 0 & 0 & \varepsilon_{yy} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_{zz} & \mu_{zx} & 0 \\ \mu_{xz} & \mu_{xx} & 0 \\ 0 & 0 & \mu_{yy} \end{bmatrix} \tag{76}$$

$$\boldsymbol{\xi} = \xi \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\zeta} = \zeta \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{77}$$

In this case, the kernel $(\overrightarrow{\mathbf{Z}} + \overleftarrow{\mathbf{Z}})$ and $(\overrightarrow{\mathbf{Y}} + \overleftarrow{\mathbf{Y}})$ commute with polynomial matrices and can be factorized in closed form [*Daniele*, 2004a]; that is to say that the closed form solution for the problem of a PEC or a PMC half plane immersed in a bianisotropic medium defined by (76) and (77) can always be obtained.

4.3. Bianisotropic Media With Vanishing $\mathbf{T}_e$ and $\mathbf{T}_h$ Matrices

[32] For appropriate values of the electromagnetic constitutive parameters, the matrices $\mathbf{T}_e(\eta)$ and $\mathbf{T}_h(\eta)$ vanish. This happens, for example, for $\boldsymbol{\xi} = 0$, $\boldsymbol{\zeta} = 0$ and $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ given as in (76). This particular case has been addressed by *Hurd and Przezdziecki* [1981] after reducing the factorization problem to a Hilbert problem. At any rate, in this case, equations (36) simplify into

$$\begin{aligned} \boldsymbol{\gamma}_V^2 - \mathbf{ZY} = 0 \\ \boldsymbol{\gamma}_I^2 - \mathbf{YZ} = 0 \end{aligned} \tag{78}$$

so that one is actually faced with the classical transmission line problem, extensively studied in the literature. In this case, the characteristic impedances are [*Paul*, 1975]

$$\overrightarrow{\mathbf{Z}} = \overleftarrow{\mathbf{Z}} = \boldsymbol{\gamma}_V^{-1}\mathbf{Z} = \boldsymbol{\gamma}_V\mathbf{Y}^{-1} = \mathbf{Y}^{-1}\boldsymbol{\gamma}_I = \mathbf{Z}\boldsymbol{\gamma}_I^{-1} \tag{79}$$

with

$$\boldsymbol{\gamma}_V = \sqrt{\mathbf{ZY}}, \quad \boldsymbol{\gamma}_I = \sqrt{\mathbf{YZ}} \tag{80}$$

By taking into account that $\boldsymbol{\gamma}_V$ commutes with the polynomial matrix $\mathbf{ZY}$, or that $\boldsymbol{\gamma}_I$ commutes with the polynomial matrix $\mathbf{YZ}$, one can express the kernel $(\overrightarrow{\mathbf{Z}} + \overleftarrow{\mathbf{Z}})$ and $(\overrightarrow{\mathbf{Y}} + \overleftarrow{\mathbf{Y}})$ in terms of a polynomial matrix multiplied by a matrix that commutes with a polynomial matrix. In this connection, once again, we remark that matrices that commute with a polynomial matrix are factorized in closed form [*Daniele*, 2004a].

4.4. Anisotropic Media of the Hurd-Przezdziecki Problem

[33] We conclude this section by briefly considering the case of the PEC half plane for $\boldsymbol{\xi} = 0$, $\boldsymbol{\zeta} = 0$, $\boldsymbol{\mu} = \mu\bar{\mathbf{I}}_3$ and

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{zz} & 0 & 0 \\ 0 & \varepsilon_{xx} & \varepsilon_{xy} \\ 0 & -\varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix} \tag{81}$$

which has been solved by *Hurd and Przezdziecki* [1985] with a different approach. In this case the $\mathbf{T}_e$ and $\mathbf{T}_h$ matrices do not vanish, and our formulation yields

$$\begin{aligned} \mathbf{t}\left(\overrightarrow{\mathbf{Z}} + \overleftarrow{\mathbf{Z}}\right)\mathbf{t}_a &= f_o(\eta)\mathbf{Q}_o(\eta) + f_1(\eta)\mathbf{Q}_1(\eta) \\ \mathbf{t}\left(\overrightarrow{\mathbf{Y}} + \overleftarrow{\mathbf{Y}}\right)\mathbf{t}_a &= g_o(\eta)S_o(\eta) + g_1(\eta)S_1(\eta) \end{aligned} \tag{82}$$

Although, for the sake of brevity, the rational matrices $\mathbf{Q}_{o,1}(\eta)$, $S_{o,1}(\eta)$ and the not-rational functions $f_{o,1}(\eta)$, $g_{o,1}(\eta)$ are not reported here, we notice that the matrices appearing in (82) commute with polynomial matrices and can certainly be factorized in closed form.

5. Numerical Factorization and Far-Field Evaluation

5.1. Normalization of the WH Equations

[34] The method to numerically factorize the WH kernel and to evaluate the far-field quantities is illustrated, for the sake of clarity, only for the simpler case of a PEC half plane. The extension of this method to deal with imperfect half planes is rather straightforward.

[35] In terms of the dimensionless normalized admittance

$$\mathbf{y}_t(\eta) = Z_o\left(\overrightarrow{\mathbf{Y}} + \overleftarrow{\mathbf{Y}}\right) \tag{83}$$

the PEC half plane problem yields (see (13) and (20))

$$\mathbf{y}_t(\eta)\mathbf{V}_+ = Z_o\mathbf{A}^s_-(\eta) + \frac{\mathbf{R}_o}{\eta - \eta_0} \tag{84}$$

where $\mathbf{R}_o = jE_0\mathbf{y}_t(\eta_0)\mathbf{e}_{0t}$ is a known term whereas $\mathbf{A}^s_-(\eta)$ is the Fourier transform of the total scattered current induced on the half plane, that is the total minus the Physical Optic (PO) current.

[36] Factorization in closed form is possible whenever $\mathbf{y}_t(\eta)$ commute with a polynomial matrix; if this does not happen, one has to use the approximate technique described by *Daniele* [2004a], which is based on the numerical solution of a Fredholm integral equation of the second kind. This technique requires that the WH matrix kernel and its inverse exist and are finite for $\eta \to \pm\infty$ on the real axis. Therefore we need to modify the kernel $\mathbf{y}_t(\eta)$ because of the following asymptotic behaviors for $\eta \to \infty$

$$\mathbf{y}_t(\eta) \approx \begin{bmatrix} a_{11}/\eta & a_{12} \\ a_{21} & \eta a_{22} \end{bmatrix} \tag{85}$$

$$\mathbf{y}_t^{-1}(\eta) \approx \begin{bmatrix} \eta b_{11} & b_{12} \\ b_{21} & b_{22}/\eta \end{bmatrix} \tag{86}$$

where a_{ij} and b_{ij} are not vanishing constants. By introducing the matrices

$$\mathbf{d}_{\substack{\ell \\ r}}(\eta) = \begin{bmatrix} \sqrt{\dfrac{k_o \pm \eta}{k_o}} & 0 \\ 0 & \sqrt{\dfrac{k_o}{k_o \pm \eta}} \end{bmatrix} \tag{87}$$

$$\mathbf{G}_e(\eta) = \mathbf{d}_\ell\, \mathbf{y}_t\, \mathbf{d}_r \tag{88}$$

equation (84) yields

$$\mathbf{G}_e(\eta)\mathbf{X}_+(\eta) = \mathbf{X}^s_-(\eta) + \frac{\mathbf{R}}{\eta - \eta_0} \tag{89}$$

with

$$\mathbf{X}_+(\eta) = \mathbf{d}_r^{-1}(\eta)\, \mathbf{V}_+(\eta) \tag{90}$$

$$\mathbf{X}_-(\eta) = Z_o\, \mathbf{d}_\ell(\eta)\, \mathbf{A}^s_-(\eta) + \frac{[\mathbf{d}_\ell(\eta) - \mathbf{d}_\ell(\eta_o)]}{\eta - \eta_0}\, \mathbf{R}_o \tag{91}$$

$$\mathbf{R} = \mathbf{d}_\ell(\eta_o)\, \mathbf{R}_o \tag{92}$$

and where $\mathbf{G}_e(\eta)$ and $\mathbf{G}_e(\eta)^{-1}$ exist and are finite for $\eta \to \pm\infty$.

[37] By introducing the normalized quantities $\overline{\eta}_o = \eta_o/k_o$ (as per equation (2)), and

$$\overline{\mathbf{G}}_e(\overline{\eta}) = \mathbf{G}_e(k_o\overline{\eta}) \tag{93}$$

$$\overline{\mathbf{X}}_+(\overline{\eta}) = k_o\mathbf{X}_+(k_o\overline{\eta}) \tag{94}$$

$$\overline{\mathbf{X}}_-(\overline{\eta}) = k_o\mathbf{X}_-(k_o\overline{\eta}) \tag{95}$$

equation (89) reduces to

$$\overline{\mathbf{G}}_e(\overline{\eta})\, \overline{\mathbf{X}}_+(\overline{\eta}) = \overline{\mathbf{X}}^s_-(\overline{\eta}) + \frac{\mathbf{R}}{\overline{\eta} - \overline{\eta}_0} \tag{96}$$

where all the involved quantities do not depend on k_o, Z_o and Y_o. That is to say that the normalized WH equation (96) does not change for a fictitious medium that has the same normalized matrices $\overline{\varepsilon}$, $\overline{\boldsymbol{\mu}}$, $\overline{\boldsymbol{\xi}}$, and $\overline{\boldsymbol{\zeta}}$, but different values of k_o, Z_o and Y_o. As a matter of fact, values of k_o with a vanishing or a very small imaginary part make the evaluation of the eigenvalues and the numerical factorization of $\mathbf{G}_e$ difficult; for this reason it is convenient to introduce a fictitious lossy medium, and in our numerical simulations we have used

$$\widetilde{k}_o = 1 - j, \quad \widetilde{Z}_o = Z_o, \quad \widetilde{Y}_o = Y_o \tag{97}$$

In the following, to avoid any confusion, the quantities computed for the fictitious medium are indicated by a tilde. The eigenvalue $\widetilde{\gamma}_i = j\widetilde{k}_o\overline{\kappa}_i$ computed for the fictitious medium permits one to evaluate the corresponding physical eigenvalue $\gamma_i = k_o\widetilde{\gamma}_i/\widetilde{k}_o$ of the real medium.

[38] In the fictitious medium, equation (89) now reads

$$\widetilde{\mathbf{G}}_e(\eta)\widetilde{\mathbf{X}}_+(\eta) = \widetilde{\mathbf{X}}^s_-(\eta) + \frac{\mathbf{R}}{\eta - \eta_0} \tag{98}$$

where numerical evaluation of $\widetilde{\mathbf{X}}_+(\eta)$ by the Fredholm method given by *Daniele* [2004a] is now possible. The physical solution $\mathbf{X}_+(\eta)$ is obtained by noticing that

$$\overline{\mathbf{X}}_+(\overline{\eta}) = k_o\mathbf{X}_+(k_o\overline{\eta}) = \widetilde{k}_o\widetilde{\mathbf{X}}_+\left(\widetilde{k}_o\overline{\eta}\right) \tag{99}$$

which yields

$$\mathbf{X}_+(\eta) = \frac{\widetilde{k}_o}{k_o}\, \widetilde{\mathbf{X}}_+\left(\widetilde{k}_o\, \eta/k_o\right) \tag{100}$$

This permits one to solve the WH problem (89) and to obtain the functions $\mathbf{V}_+(\eta)$, $\mathbf{I}_a(\eta) = \overrightarrow{\mathbf{Y}}\mathbf{V}_+$, and $\mathbf{I}_b(\eta) = \overleftarrow{\mathbf{Y}}\mathbf{V}_+$.

5.2. Field Evaluation by Inverse Fourier Transformation

[39] In the following, without loss of generality, we consider only the evaluation of the transverse component of the electric field for the PEC half plane problem where, to simplify the notation, we set $\overleftarrow{\gamma}_V = \overleftarrow{\gamma}$, $\overrightarrow{\gamma}_V = \overrightarrow{\gamma}$. The transverse component of the magnetic field can be similarly obtained. The longitudinal field components $E_y(z, x, y)$ and $H_y(z, x, y)$ are obtained from the transverse ones by Maxwell's equations [see *Daniele*, 2006].

[40] The plus function $\mathbf{V}_+(\eta)$ is obtained by solving equation (84). The transverse field $\mathbf{E}_t(z, y, x)$ is then obtained by inverse Fourier transformation

$$\mathbf{E}_t = \mathbf{E}_t^p + \frac{\exp(-j\alpha_o z)}{2\pi}\int_{B_+}\exp(-j\eta x - \overline{\gamma}y) \cdot \mathbf{V}_+(\eta)\times\hat{\mathbf{y}}\,d\eta \tag{101}$$

with $\overline{\gamma} = \overrightarrow{\gamma}$ for $y > 0$, and $\overline{\gamma} = \overleftarrow{\gamma}$ in the region $y < 0$, and where the integration path B_+ is a horizontal straight line located above all the singularities of $\mathbf{V}_+(\eta)$. The primary field $\mathbf{E}_t^p(z, y, x) = \mathbf{E}_t^i(z, y, x) + \mathbf{E}_t^r(z, y, x)$ represents the contribution of the incident plus the reflected field; the primary field is evaluated by assuming an entire PEC plane at $y = 0$. Notice that $\mathbf{E}_t^p$ is equal to zero for $y < 0$ if the incident wave impinging of the half plane propagates in the negative y direction. Vice versa, the primary field is zero for $y > 0$ whenever the incident wave propagates in the positive y direction.

[41] Let us now assume that the incident plane wave propagates in the negative y direction. By using the Cayley representation for the exponential factor $\exp(-\overline{\gamma}y)$ in (101) one gets

$$\begin{aligned}\hat{\mathbf{y}}\times\mathbf{E}_t &= \hat{\mathbf{y}}\times\mathbf{E}_t^p + \frac{e^{-j\alpha_o z}}{2\pi}\int_{B_+}\frac{\gamma_2\mathbf{1}-\overrightarrow{\gamma}}{\gamma_2-\gamma_1} \\ &\quad\cdot\mathbf{V}_+(\eta)\,e^{(-j\eta x-\gamma_1 y)}d\eta + \frac{e^{-j\alpha_o z}}{2\pi}\int_{B_+}\frac{\gamma_1\mathbf{1}-\overrightarrow{\gamma}}{\gamma_1-\gamma_2} \\ &\quad\cdot\mathbf{V}_+(\eta)\,e^{(-j\eta x-\gamma_2 y)}d\eta\end{aligned} \tag{102}$$

for $y > 0$, and

$$\begin{aligned}\hat{\mathbf{y}} = &\frac{e^{-j\alpha_o z}}{2\pi}\int_{B_+}\frac{\gamma_4\mathbf{1}-\overleftarrow{\gamma}}{\gamma_4-\gamma_3}\mathbf{V}_+(\eta)\,e^{(-j\eta x-\gamma_3 y)}d\eta \\ &+\frac{e^{-j\alpha_o z}}{2\pi}\int_{B_+}\frac{\gamma_3\mathbf{1}-\overleftarrow{\gamma}}{\gamma_3-\gamma_4}\mathbf{V}_+(\eta)\,e^{(-j\eta x-\gamma_4 y)}d\eta\end{aligned} \tag{103}$$

for $y < 0$. By further assuming an incident plane wave with a propagation factor equal to $\exp[-j\alpha_o z - j\eta_o x - \gamma_3(\eta_o)y]$ (without loss of generality, the case of an incident plane wave with propagation factor equal to $\exp[-j\alpha_o z - j\eta_o x - \gamma_4(\eta_o)y]$ can be omitted) one gets

$$\hat{\mathbf{y}}\times\mathbf{E}_t^i = E_0\,\mathbf{e}_{0t}\exp[-j\alpha_o z - j\eta_o x - \gamma_3(\eta_o)y] \tag{104}$$

$$\begin{aligned}\hat{\mathbf{y}}\times\mathbf{E}_t^r = &\exp(-j\alpha_o z - j\eta_o x) \\ &\cdot\left\{\mathbf{E}_{t1}^r\,e^{-\gamma_1(\eta_o)y} + \mathbf{E}_{t2}^r\,e^{-\gamma_2(\eta_o)y}\right\}\end{aligned} \tag{105}$$

where $\mathbf{e}_{0t}$ is the eigenvector of $\overleftarrow{\gamma}(\eta_o)$ associated with the eigenvalue $\gamma_3(\eta_o)$, whereas the amplitude vectors $\mathbf{E}_{t1}^r$ and $\mathbf{E}_{t2}^r$ can be obtained by geometrical optics considerations. For example, while dealing with a PEC plane, one has $\mathbf{E}_{t1}^r = c_1\mathbf{e}_{t1}$ and $\mathbf{E}_{t2}^r = c_2\mathbf{e}_{t2}$, where $\mathbf{e}_{t1}$ and $\mathbf{e}_{t2}$ are the eigenvectors of $\overrightarrow{\gamma}(\eta_o)$ associated with the eigenvalue $\gamma_1(\eta_o)$ and $\gamma_2(\eta_o)$; the scalar coefficients c_1 and c_2 are simply obtained by enforcing $\mathbf{E}_t^p = 0$ on the PEC plane.

[42] The far-field contributions are evaluated by applying the saddle point method [*Felsen and Marcuvitz*, 1973] to each integral of (102) and (103). For each given observation point with azimuthal angle ϕ, this method requires the determination of the saddle points $\overline{\eta}_s$ of the function

$$q(\overline{\eta}) = \overline{\eta}\cos\phi + \overline{\kappa}(\overline{\eta})\sin\phi \tag{106}$$

with

$$\overline{\kappa}(\overline{\eta}) = -j\gamma_i(\overline{\eta}), \quad i = 1, 4 \tag{107}$$

and the determination of the steepest descent paths (SDP) that cross the saddle points. In fact, the integration path B_+ of (102) and (103) is warped into a SDP, and each saddle point has is own SDP. Occasionally, more saddle points may occur for i = 1, 2, 3, or 4; in these cases, application of the saddle point method is more difficult. For the sake of simplicity we do not discuss these cases, and we assume that each integral has only one significant saddle point. We also observe that the $\overline{\eta}_s$ are real for a lossless medium. Since the observation angle ϕ appearing in (106) depends on the location of the observation point, it may happen that the SDP captures the pole $\eta = \eta_o$ of the function $\mathbf{V}_+$. A detailed study, not reported here, shows that the pole is captured only if $\overline{\eta}_s > \overline{\eta}_o$; when this happens, the pole $\eta = \eta_o$ is always captured clockwise. A careful study of the residues of the integrands and use of the boundary condition for the PEC half plane permit one to write, for $y > 0$

$$\begin{aligned}\mathbf{E}_t &= \mathbf{E}_t^p - \frac{e^{-j(\alpha_o z+\eta_o x)}}{2\pi} E_0 \Big[u(\overline{\eta}_{s1} - \overline{\eta}_o)\, \mathbf{E}_{t1}^r\, e^{-\gamma_1(\eta_o) y} \\ &\quad + u(\overline{\eta}_{s2} - \overline{\eta}_o)\, \mathbf{E}_{t2}^r\, e^{-\gamma_2(\eta_o) y} \Big] + \frac{e^{-j\alpha_o z}}{2\pi} \\ &\quad \cdot \int_{\mathrm{SDP}_1} \frac{\gamma_2 \mathbf{1} - \overrightarrow{\gamma}}{\gamma_2 - \gamma_1} \mathbf{V}_+(\eta) \times \hat{\mathbf{y}}\, e^{-j\eta x - \gamma_1 y}\, \mathrm{d}\eta \\ &\quad + \frac{e^{-j\alpha_o z}}{2\pi} \int_{\mathrm{SDP}_2} \frac{\gamma_1 \mathbf{1} - \overrightarrow{\gamma}}{\gamma_1 - \gamma_2} \mathbf{V}_+(\eta) \times \hat{\mathbf{y}}\, e^{-j\eta x - \gamma_2 y}\, \mathrm{d}\eta \end{aligned} \quad (108)$$

whereas for $y < 0$, one gets

$$\begin{aligned}\mathbf{E}_t &= u(\overline{\eta}_{s3} - \overline{\eta}_o)\, \mathbf{E}_t^i + \frac{e^{-j\alpha_o z}}{2\pi} \int_{\mathrm{SDP}_3} \frac{\gamma_4 \mathbf{1} - \overleftarrow{\gamma}}{\gamma_4 - \gamma_3} \\ &\quad \cdot \mathbf{V}_+(\eta) \times \hat{\mathbf{y}}\, e^{-j\eta x - \gamma_3 y}\, \mathrm{d}\eta + \frac{e^{-j\alpha_o z}}{2\pi} \int_{\mathrm{SDP}_4} \frac{\gamma_3 \mathbf{1} - \overleftarrow{\gamma}}{\gamma_3 - \gamma_4} \\ &\quad \cdot \mathbf{V}_+(\eta) \times \hat{\mathbf{y}}\, e^{-j\eta x - \gamma_4 y} \mathrm{d}\eta \end{aligned} \quad (109)$$

The previous results show that the total field does not present any reflected wave contribution in the region $y > 0$ if the saddle points $\overline{\eta}_{si}$ satisfy the condition $\overline{\eta}_{si} > \overline{\eta}_o$, for $i = 1, 2$. Furthermore, for $y < 0$, the total field does not present any geometrical optics contribution (in particular, the incident wave contribution) if the saddle point $\overline{\eta}_{s3}$ satisfies the condition $\overline{\eta}_{s3} < \overline{\eta}_o$.

[43] As far as the evaluation of the SDP integrals is concerned, after some algebraic manipulations one may prove that all the transverse field components, including the magnetic field ones, can be expressed in terms of scalar integrals of the form

$$\mathcal{I}_s = \frac{k_o}{2\pi} \int_{\mathrm{SDP}} A(k_o \overline{\eta}) \exp\left(-jk_o \rho\, q\right) \mathrm{d}\overline{\eta} \quad (110)$$

where ρ is the radial distance from the observation point to the edge of the half plane, and q is given in (106). The saddle points $\overline{\eta}_s$ are obtained by evaluating the zeroes of the derivative of the phase term

$$\frac{\mathrm{d}q}{\mathrm{d}\overline{\eta}} = \cos\phi + \frac{\mathrm{d}\overline{\kappa}}{\mathrm{d}\overline{\eta}} \sin\phi \quad (111)$$

with

$$\frac{\mathrm{d}\overline{\kappa}}{\mathrm{d}\overline{\eta}} = -\frac{\frac{\partial f(\overline{\eta},\overline{\kappa})}{\partial \overline{\eta}}}{\frac{\partial f(\overline{\eta},\overline{\kappa})}{\partial \overline{\kappa}}} \quad (112)$$

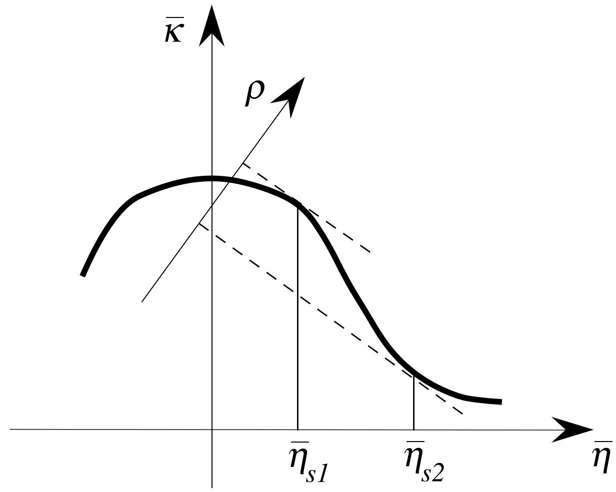

Figure 3. Geometrical evaluation of the saddle points for lossless media.

The saddle points are therefore given by the solutions, with respect to $\overline{\eta}$, of the following system of algebraic equations

$$\begin{aligned} & f(\overline{\eta},\overline{\kappa}) = 0 \\ & \frac{\partial f(\overline{\eta},\overline{\kappa})}{\partial \overline{\kappa}} \cos\phi - \frac{\partial f(\overline{\eta},\overline{\kappa})}{\partial \overline{\eta}} \sin\phi = 0 \end{aligned} \quad (113)$$

which does not change by changing ϕ into $(\phi - \pi)$. As a matter of fact, for lossless media, the saddle points can be obtained graphically [*Felsen and Marcuvitz*, 1973, p. 110] by considering the dispersion curves $f(\overline{\eta}, \overline{\kappa})$ and the vector distance $\boldsymbol{\rho} = \rho(\hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi)$ from the observation point to the edge of the half plane, as shown in Figure 3. In Figure 3, at the saddle points $\overline{\eta}_{s1}$ and $\overline{\eta}_{s2}$, the straight lines tangent to the dispersion curve $\overline{\kappa}(\overline{\eta})$ are orthogonal to $\boldsymbol{\rho}$, and this happens only and for all the saddle points.

[44] More generally, since equations (113) are algebraic, the saddle points $\overline{\eta}_s$ are defined to be the zeroes of the polynomial in $\overline{\eta}$ obtained from the resultant

$$R\left[f(\overline{\eta},\overline{\kappa}), \frac{\partial f(\overline{\eta},\overline{\kappa})}{\partial \overline{\kappa}} \cos\phi - \frac{\partial f(\overline{\eta},\overline{\kappa})}{\partial \overline{\eta}} \sin\phi, \overline{\kappa} \right] \quad (114)$$

of the two equations (113) with respect to $\overline{\kappa}$ [*Bliss*, 2004].

[45] For lossless media, the contribution of each real saddle point to the integral (110) is

$$\begin{aligned}\mathcal{I}_s &= \frac{A(k_o \overline{\eta}_s) \exp(\mp j\pi/4)}{\sqrt{2\pi k_o \rho |q''(\overline{\eta}_s)|}} \\ &\quad \cdot \exp\left\{-jk_o \rho [\overline{\eta}_s \cos\phi + \overline{\kappa}(\overline{\eta}_s) \sin\phi]\right\} \end{aligned} \quad (115)$$

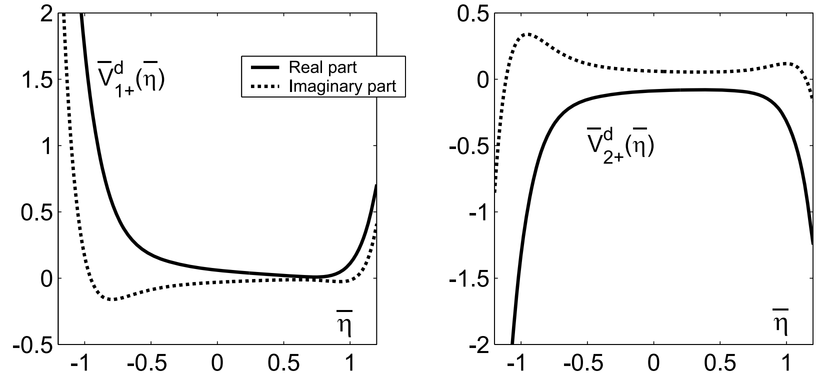

Figure 4. Real and imaginary part of the function (left) $V^{d}_{1+}(\overline{\eta})$ and (right) $V^{d}_{2+}(\overline{\eta})$, given in (116), for $\overline{\alpha}_o = \overline{\eta}_o = 1/2$.

where the plus or minus sign in the exponential factor is chosen according to the sign of $q''\overline{\eta}_s$) [*Felsen and Marcuvitz*, 1973, p. 387].

[46] In this connection we recall that the function $\mathbf{V}_+(\eta)$ has one pole in $\eta = \eta_o$, with residue $\mathbf{T} = jE_o\mathbf{e}_{0t}$. The function $\mathbf{T}/(\eta - \eta_o)$ is the Fourier transform of the geometrical optic field over the aperture ($x > 0$, $y = 0$). The Physical Optics (PO) contribution to the diffracted field is calculated by approximating $A(k_o\overline{\eta})$ with the value obtained by using $\mathbf{V}_+^{PO} = \mathbf{T}/(\eta - \eta_o)$, $\mathbf{I}_a^{PO} = \overleftrightarrow{\mathbf{Y}}\ \mathbf{T}/(\eta - \eta_o)$, or $\mathbf{I}_b^{PO} = \overleftarrow{\mathbf{Y}}\ \mathbf{T}/(\eta - \eta_o)$ in (108) and (109). For this reason, the quantity

$$\mathbf{V}_+^{d}(\eta) = \mathbf{V}_+(\eta) - \frac{\mathbf{T}}{\eta - \eta_o} \tag{116}$$

evaluated at the saddle points is able to represent the difference between the true diffracted field and the PO contribution to the diffracted field. Notice that the superscript d in (116) stays for *difference*. This difference is considered in the numerical case study that follows.

5.3. Numerical Results for a PEC Half Plane in a Gyrotropic Medium

[47] Let us consider a lossless gyrotropic medium with $\overline{\boldsymbol{\mu}} = \mathbf{1}$, $\overline{\boldsymbol{\xi}} = \overline{\boldsymbol{\zeta}} = \mathbf{0}$ and

$$\overline{\varepsilon} = \begin{bmatrix} 2 & j2/3 & j/4 \\ -j2/3 & 3 & j/2 \\ -j/4 & -j/2 & 3/2 \end{bmatrix} \tag{117}$$

By choosing $\overline{\alpha}_o = 1/2$, from equations (48) and (56), we obtain

$$f(\overline{\eta}, \overline{\kappa}) = \overline{\kappa}^4 + \overline{a}\,\overline{\kappa}^3 + \overline{b}\,\overline{\kappa}^2 + \overline{c}\,\overline{\kappa} + \overline{d} \tag{118}$$

with

$$\begin{bmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \\ \overline{d} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{101}{24} - 3\overline{\eta}^2 \\ j\dfrac{2}{9}(1 - \overline{\eta}) \\ \dfrac{3253}{864} + \dfrac{\overline{\eta}}{12} - \dfrac{154}{27}\,\overline{\eta}^2 + \overline{\eta}^4 \end{bmatrix} \tag{119}$$

These data yield 12 normalized branch points in the normalized $\overline{\eta}$ plane; four of them are real

$$\{\overline{\eta}_{1_{\rm br}}, \overline{\eta}_{2_{\rm br}}, \overline{\eta}_{3_{\rm br}}, \overline{\eta}_{4_{\rm br}}\} = \{-1.38078, -1.00638, 1.04821, 1.31776\} \tag{120}$$

and eight are complex

$$\left\{ \begin{array}{ll} -1.4998 \pm j0.7909, & -0.8605 \pm j0.1128, \\ 1.1770 \pm j0.4836, & 1.1939 \pm j0.2892 \end{array} \right\}$$

[48] The $\overline{\kappa}$ values at the four real branch points are

$$\{\overline{\kappa}_{1_{\rm br}}, \overline{\kappa}_{2_{\rm br}}, \overline{\kappa}_{3_{\rm br}}, \overline{\kappa}_{4_{\rm br}}\} = \{0.168685, -0.205354, 0.00587302, -0.0351789\} \tag{121}$$

Recall that the real branch points individuate the four branch points of the characteristic impedance and admittance. The dispersion curves for this case are shown in Figure 2, where it is evident that two different kinds of waves can propagate in this gyrotropic medium.

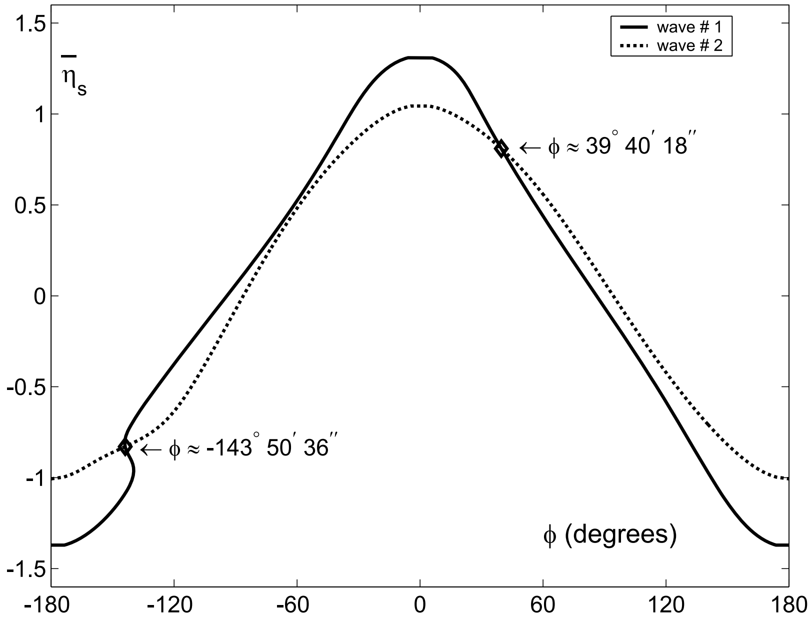

Figure 5. Saddle points η_s in the observation angular region $\{-180° < \phi \leq 180°\}$ for the first and second wave of the dispersion diagram of Figure 2, in case of $\overline{\alpha}_o = 1/2$. The two values of ϕ where η_s is the same for the first and second wave are reported. Notice that the first wave has two or three saddle points in the region around $\phi = -140°$.

[49] We now consider an incident plane wave of the second kind (the wave 2 in the dispersion diagram of Figure 2), with $E_o = 1$, $\phi_i = -3\pi/4$, and

$$\begin{aligned} \alpha_o = k_o/2\,, \quad & \eta_o = -k_o \cos\phi_i\,, \\ \overline{\alpha}_o = 1/2\,, \quad & \overline{\eta}_o = -\cos\phi_i \end{aligned} \tag{122}$$

that is for $\widetilde{\alpha}_o = \widetilde{k}_o \cos\beta_i$, $\widetilde{\eta}_o = -\widetilde{k}_o \cos\phi_i$, and $\beta_i = \pi/3$. In the free space, and not in the gyrotropic medium, this wave would correspond to a plane wave incident with an azimuthal angle ϕ_i, and a zenithal angle β_i. The polarization vector in the gyrotropic medium

$$\mathbf{e}_{ot} = \begin{bmatrix} -0.0798 - j0.5388 \\ 0.8387 \end{bmatrix} \tag{123}$$

does not depend on the value of $\widetilde{k}_o$.

[50] The incident wave originates two diffracted waves, one of the first and one of the second kind, which are computed very accurately by numerically solving, with the technique explained by *Daniele* [2004a], the Fredholm equation associated with the factorization of the WH kernel. The results of Figure 4 show the real and the imaginary part of the two normalized components of the quantity $\overline{\mathbf{V}}_+^d(\overline{\eta}) = k_o \mathbf{V}_+^d(\overline{\eta})$ given in (116). As previously explained, in order to obtain the far diffracted field according to (115) it is enough to know the saddle points and the value of $\mathbf{V}_+^d(\eta)$ at the saddle points. The location η_s of the saddle points is obtained from (114), and the values of η_s in the observation angular region $\{-180° < \phi \leq 180°\}$ are reported in Figure 5. Tables 1 and 2 report the values of the diffracted field components

$$E^s_{x,z} = \frac{E^{PO}_{x,z} + E^d_{x,z}}{\sqrt{k_o \rho}} \exp\left(-j\,\theta_c\, k_o \rho\right) \exp\left(-j\alpha_o z\right) \tag{124}$$

Table 1. Field Components for the First Wave Diffracted Field at Four Different Observation Angles ϕ, in Case of $\overline{\alpha}_o = \overline{\eta}_o = 1/2$[a]

ϕ	E_x^{PO}	E_x^d	E_z^{PO}	E_z^d	θ_c
45°	0.1232 − j0.0448	0.0029 + j0.0218	0.0365 − j0.2505	−0.0038 − j0.0123	1.50575
135°	−0.0182 + j0.0163	−0.0633 − j0.0239	−0.0172 + j0.0259	0.0298 − j0.1199	1.60743
−45°	−0.4736 − j3.3692	−0.0230 − j0.0271	−5.1562 + j0.0384	0.0504 − j0.2197	1.46782
−135°	0.1488 + j1.4465	1.4096 − j2.9930	2.2875 − j0.0757	1.0800 − j2.3424	1.34645

[a]See equation (124).

Table 2. Field Components for the Second Wave Diffracted Field at Four Different Observation Angles ϕ, in Case of $\overline{\alpha}_o = \overline{\eta}_o = 1/2$[a]

ϕ	E_x^{PO}	E_x^d	E_z^{PO}	E_z^d	θ_c
45°	−0.8191 + j0.1432	−0.0109 − j0.0201	−0.0302 + j1.3237	−0.0567 + j0.0019	1.09735
135°	0.2144 − j0.0396	−0.0848 − j0.0754	0.0087 − j0.3219	−0.1371 + j0.0497	1.00051
−45°	0.3280 + j2.3936	0.0332 + j0.0337	3.6851 − j0.0347	−0.0640 + j0.2699	1.12522
−135°	−0.2353 − j1.7434	−0.4665 + j1.6206	−2.7566 + j0.0255	−0.3959 + j1.3621	1.19875

[a]See equation (124).

relative to the first and second wave for different observation angles. The values for the first kind wave are given in Table 1, whereas Table 2 shows the values relative to the wave of the second kind. Notice from equation (124) that in general, the phase of the cylindrical waves 1 and 2, with respect to the phase one has for an isotropic medium, is corrected by a θ_c factor which depends on the azimuthal observation angle ϕ as well as on the kind (1 or 2) of the diffracted wave.

6. Conclusion

[51] A very general Wiener-Hopf approach to study the electromagnetic diffraction by an imperfect half plane immersed in a linear homogeneous bianisotropic medium is presented. The effects of the material are summarized by introducing characteristic impedance and admittance matrices, which indeed allow for a straightforward formulation of the Wiener-Hopf problem.

[52] In the simpler case of perfect electric conducting and perfect magnetic conducting half planes, the Wiener-Hopf equations involve matrices of order 2, which are factorized in closed form for special form of the material constitutive tensors. Four of these special cases are discussed in detail.

[53] The superiority of the Wiener-Hopf technique with respect to other existing techniques is rather evident when one deals with the most general problem, where the Wiener-Hopf matrix kernels must be factorized numerically by using a technique presented in this paper. Our numerical approach is discussed in detail on one example, by considering the previously unsolved problem of a perfect electric conducting half plane immersed in a gyrotropic medium. Numerical results are reported to show that the diffracted field contribution can be obtained by the saddle point integration method.

Appendix A

[54] For the imperfect half plane problem, by taking into account the boundary conditions (6), the arrays

$$\begin{cases} \mathbf{F}_{1+} = \mathbf{V}_a + \mathbf{Z}_a\mathbf{I}_a - \mathbf{Z}_{ab}\mathbf{I}_b \\ \mathbf{F}_{2+} = \mathbf{V}_b + \mathbf{Z}_b\mathbf{I}_b - \mathbf{Z}_{ba}\mathbf{I}_a \end{cases} \tag{A1}$$

are plus functions of the complex variable η, since they are Fourier transforms of functions that vanish for $x < 0$. Furthermore, by taking into account the continuity of the electromagnetic field on the aperture half plane region $\{x > 0, y = 0\}$, it is straightforward to prove that the arrays

$$\begin{cases} \mathbf{X}_{1-} = \mathbf{V}_a - \mathbf{V}_b \\ \mathbf{X}_{2-} = \mathbf{I}_a + \mathbf{I}_b \end{cases} \tag{A2}$$

are minus functions of the complex variable η, since they are obtained by Fourier transforming functions that vanish for $x > 0$. By eliminating $\mathbf{V}_a$, $\mathbf{V}_b$, $\mathbf{I}_a$, and $\mathbf{I}_b$ from (A1) and (A2) and from the first equation of (11) and (12), one gets

$$\mathbf{G}(\eta)\begin{bmatrix}\mathbf{F}_{1+}\\ \mathbf{F}_{2+}\end{bmatrix} = \begin{bmatrix}\mathbf{X}_{1-}\\ \mathbf{X}_{2-}\end{bmatrix} \tag{A3}$$

with

$$\mathbf{G}(\eta) = \begin{bmatrix} \left(\overset{\rightarrow}{\mathbf{Z}} + \mathbf{Z}_a + \mathbf{Z}_{ab}\right)\overset{\Leftrightarrow}{\mathbf{Y}}\left[\left(\overset{\rightarrow}{\mathbf{Z}} + \mathbf{Z}_a\right)\overset{\Leftrightarrow}{\mathbf{Y}}\overset{\leftarrow}{\mathbf{Z}} - \mathbf{Z}_{ab}\overset{\Leftrightarrow}{\mathbf{Y}}\overset{\rightarrow}{\mathbf{Z}}\right] \\ -\left(\overset{\leftarrow}{\mathbf{Z}} + \mathbf{Z}_b + \mathbf{Z}_{ba}\right)\overset{\Leftrightarrow}{\mathbf{Y}}\left[\left(\overset{\leftarrow}{\mathbf{Z}} + \mathbf{Z}_b\right)\overset{\Leftrightarrow}{\mathbf{Y}}\overset{\rightarrow}{\mathbf{Z}} - \mathbf{Z}_{ba}\overset{\Leftrightarrow}{\mathbf{Y}}\overset{\leftarrow}{\mathbf{Z}}\right] \end{bmatrix}^{-1} \tag{A4}$$

$$\overset{\Leftrightarrow}{\mathbf{Y}} = \left(\overset{\rightarrow}{\mathbf{Z}} + \overset{\leftarrow}{\mathbf{Z}}\right)^{-1} \tag{A5}$$

$$\begin{bmatrix}\mathbf{X}_{1-}\\ \mathbf{X}_{2-}\end{bmatrix} = \begin{bmatrix}\mathbf{X}^s_{1-}\\ \mathbf{X}^s_{2-}\end{bmatrix} + \frac{jE_o\mathbf{G}(\eta_o)}{\eta - \eta_o}\begin{bmatrix}\mathbf{e}_{0t} + (\mathbf{Z}_a + \mathbf{Z}_{ab})\mathbf{h}_{0t}\\ \mathbf{e}_{0t} - (\mathbf{Z}_b + \mathbf{Z}_{ba})\mathbf{h}_{0t}\end{bmatrix} \tag{A6}$$

The voltage and current functions in terms of the minus WH unknowns are

$$\begin{bmatrix}\mathbf{V}_a\\ \mathbf{I}_a\end{bmatrix} = \begin{bmatrix}\overset{\rightarrow}{\mathbf{Z}}\overset{\Leftrightarrow}{\mathbf{Y}}\\ \overset{\Leftrightarrow}{\mathbf{Y}}\end{bmatrix}\left(\overset{\leftarrow}{\mathbf{Z}}\,\mathbf{X}_{2-} + \mathbf{X}_{1-}\right) \tag{A7}$$

$$\begin{bmatrix}\mathbf{V}_b\\ \mathbf{I}_b\end{bmatrix} = \begin{bmatrix}\overset{\leftarrow}{\mathbf{Z}}\overset{\Leftrightarrow}{\mathbf{Y}}\\ \overset{\Leftrightarrow}{\mathbf{Y}}\end{bmatrix}\left(\overset{\rightarrow}{\mathbf{Z}}\,\mathbf{X}_{2-} - \mathbf{X}_{1-}\right) \tag{A8}$$

Appendix B

[55] The (4 × 4) matrix **P** given in (26) is obtained by erasing the vanishing third and sixth rows and third and sixth columns of the (6 × 6) matrix

$$\mathbf{P}_t = -\mathbf{\Gamma}_0\mathbf{\Gamma}_y\left[\overline{\overline{\mathbf{1}}}_t\left(\mathbf{\Gamma}_t - \mathbf{W}_{ty}\right)\widehat{\mathbf{W}}_y\left(\overline{\overline{\mathbf{1}}}_y\mathbf{\Gamma}_t - \mathbf{W}_{yt}\right) - \mathbf{W}_{tt}\right]\mathbf{\Gamma}_0^t \tag{B1}$$

where $\mathbf{\Gamma}_0^t$ is the transpose of $\mathbf{\Gamma}_0$, and where we have introduced the following (6 × 6) matrices

$$\overline{\overline{\mathbf{1}}}_t = \begin{bmatrix} \hat{\mathbf{z}}\hat{\mathbf{z}} + \hat{\mathbf{x}}\hat{\mathbf{x}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{z}}\hat{\mathbf{z}} + \hat{\mathbf{x}}\hat{\mathbf{x}} \end{bmatrix}_{zxyzxy} \tag{B2}$$

$$\overline{\overline{\mathbf{1}}}_y = \begin{bmatrix} \hat{\mathbf{y}}\hat{\mathbf{y}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{y}}\hat{\mathbf{y}} \end{bmatrix}_{zxyzxy} \tag{B3}$$

$$\mathbf{\Gamma}_t = -j\begin{bmatrix} \mathbf{0} & \overline{\mathbf{1}} \times (\alpha_0\hat{\mathbf{z}} + \eta\hat{\mathbf{x}}) \\ \overline{\mathbf{1}} \times (\alpha_0\hat{\mathbf{z}} + \eta\hat{\mathbf{x}}) & \mathbf{0} \end{bmatrix}_{zxyzxy} \tag{B4}$$

$$\mathbf{\Gamma}_y = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{y}} \times \overline{\mathbf{1}} \\ \hat{\mathbf{y}} \times \overline{\mathbf{1}} & \mathbf{0} \end{bmatrix}_{zxyzxy} \tag{B5}$$

$$\mathbf{\Gamma}_0 = \begin{bmatrix} \hat{\mathbf{y}} \times \overline{\mathbf{1}} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{1}}_t \end{bmatrix}_{zxyzxy} \tag{B6}$$

$$\mathbf{W} = j\omega\begin{bmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ -\boldsymbol{\zeta} & -\boldsymbol{\mu} \end{bmatrix}_{zxyzxy} \tag{B7}$$

$$\widehat{\mathbf{W}}_y = \frac{-j}{\omega\left(\varepsilon_y\mu_y - \xi_y\zeta_y\right)}\begin{bmatrix} \mu_y\hat{\mathbf{y}}\hat{\mathbf{y}} & \xi_y\hat{\mathbf{y}}\hat{\mathbf{y}} \\ -\zeta_y\hat{\mathbf{y}}\hat{\mathbf{y}} & -\varepsilon_y\hat{\mathbf{y}}\hat{\mathbf{y}} \end{bmatrix}_{zxyzxy} \tag{B8}$$

$$\mathbf{W}_{tt} = \overline{\overline{\mathbf{1}}}_t\mathbf{W}\overline{\overline{\mathbf{1}}}_t \tag{B9}$$

$$\mathbf{W}_{ty} = \overline{\overline{\mathbf{1}}}_t\mathbf{W}\overline{\overline{\mathbf{1}}}_y \tag{B10}$$

$$\mathbf{W}_{yt} = \overline{\overline{\mathbf{1}}}_y\mathbf{W}\overline{\overline{\mathbf{1}}}_t \tag{B11}$$

$$\mathbf{W}_{yy} = \overline{\overline{\mathbf{1}}}_y\mathbf{W}\overline{\overline{\mathbf{1}}}_y \tag{B12}$$

and the following (3 × 3) matrices

$$\overline{\mathbf{1}} = \overline{\mathbf{1}}_3 = [\hat{\mathbf{z}}\hat{\mathbf{z}} + \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}]_{zxy} \tag{B13}$$

$$\overline{\mathbf{1}}_t = [\hat{\mathbf{z}}\hat{\mathbf{z}} + \hat{\mathbf{x}}\hat{\mathbf{x}}]_{zxy} \tag{B14}$$

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